

# Energy Dissipation by Friction in Dynamic Multibody Contact Problems

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**Abstract** The bi-potential method has been successfully applied for the modeling of frictional contact problems in static cases. This paper presents the application of this method for dynamic analysis of impact problems with multiple deformable bodies. Instead of second order algorithms, a first order algorithm is applied for the numerical integration of the time-discretized equation of motion. The numerical results show clearly the physical energy dissipation introduced by frictional effects between the solids in contact.

**Key words:** Impact, Energy dissipation, Bi-potential method, Time-integration

## INTRODUCTION

Problems involving contact and friction are among the most difficult ones in mechanics and at the same time of crucial practical importance in many engineering branches. The main mathematical difficulty lies in the severe contact non-linearities because the natural first order constitutive laws of contact and friction phenomena are expressed by non-smooth multivalued force-displacement or force-velocity relations. In the past decade, substantial progress has been made in the analysis of contact problems using finite element procedures. A large number of algorithms for the numerical solution of the related finite element equations and inequalities have been presented in the literature. Review papers may be consulted for an extensive list of references [1-3]. See also the monographs by Kikuchi and Oden [4], Zhong [5] and Wriggers [6]. The popular penalty approximation [7] and 'mixed' or 'trial-and-error' methods appear, at first glance, suitable for many applications. But in this kind of method, the contact boundary conditions and friction laws are not satisfied accurately and it is tricky for the users to choose appropriate penalty factors. They may fail for stiff problems because of unpleasant numerical oscillations between contact statuses. The augmented Lagrangian method first appeared to deal with constrained minimization problems. Since friction problems are not minimization problems, the formulation needs to be extended. Alart & Curnier [8], Simo & Laursen [9] and De Saxcé & Feng [10] have obtained some extensions in mutually independent works. The first two works consist of applying Newton's method to the saddle-point equations of the augmented Lagrangian. De Saxcé & Feng proposed a theory called ISM (Implicit Standard Materials) and a bi-potential method, in which another augmented Lagrangian formulation was developed, which is essentially different from that of the first two works. In particular, in the bi-potential method, the frictional contact problem is treated in a reduced system by means of a reliable and efficient predictor-corrector solution algorithm. For the unilateral contact problems with friction, the classic approach is based on two minimum principles or two variational inequalities: the first for unilateral contact and the second for friction. The bi-potential method leads to a single displacement variational principle and a unique inequality. In consequence, the unilateral contact and the friction are coupled via a contact bi-potential. The application of the augmented Lagrangian method to the contact laws leads to an equation of projection onto Coulomb's cone, strictly equivalent to the original inequality [11]. For additional comments, see also the interesting discussion by Klarbring *et al.* [12, 13]. For dynamic implicit analysis in structural mechanics, the most commonly used integration algorithm is the second order algorithm such as Newmark, Wilson, etc. Wriggers *et al.* [14] have developed a radial

return mapping scheme to deal with impact-contact problems. Armero & Petocz [15] and Laursen & Chawla [16] have considered frictionless dynamic impact under the auspices of a conservative system, and have proposed the means to address the dynamic contact conditions so that they preserve the global conservation properties. Armero & Petocz [15] presume a penalty enforcement of the contact constraint, algorithmically preserving the energy dissipation associated with a new contact event and restoring it to the system upon release. Laursen & Chawla [16] also choose to concede an interpenetration of the contact surfaces in order to establish a Lagrange multiplier solution for each contact time step with energy conservation. In both cases, the contact constraints have been modified in pursuit of the conservation properties, resulting in an incomplete enforcement of what might be considered normal geometric constraints (i.e. impenetrability). Recently, Laursen & Love [17] have proposed an improved implicit integration scheme with a velocity update algorithm to avoid the interpenetration of the contact surfaces. It is well known that in impact problems, the velocity and acceleration are not continuous because of sudden changes in contact conditions (impact, release of contact). So the second order algorithms with regularity constraints may lead to serious errors. To avoid this shortcoming, some first order algorithms have been proposed by Zienkiewicz *et al.* [18], Jean [19]. The aim of the present paper is to apply the bi-potential method for contact modeling in dynamic cases in the field of Non-Smooth Dynamics using the first order algorithm for integration of the equation of motion. The algorithm developed is implemented into the finite element code FER/Contact, using C++ with object oriented programming techniques. Two numerical examples are performed in this study to show the validity of the model developed. The first example concerns the oblique impact of a 2D elastic plate onto a rigid surface with rebounding. The second example simulates the impact of two elastic cylinders in rigid walls. In order to show the physical energy dissipation by frictional effects, frictionless and frictional contact are considered for the second example.

## 1. The bi-potential method

### *Unilateral contact and Coulomb's friction laws*

First of all, some basic definitions and notations are set up. Let  $\Omega_1$  and  $\Omega_2$  be two bodies in contact at a point M for some value of the time. The instantaneous velocity of the particles of  $\Omega_1$  and  $\Omega_2$  passing at point M being, respectively,  $\dot{\mathbf{u}}_1$  and  $\dot{\mathbf{u}}_2$ , where the superposed dot denotes the time derivative. The relative velocity is  $\dot{\mathbf{u}} = \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2$ . Let  $\mathbf{r}$  be the contact reaction acting at M from  $\Omega_2$  onto  $\Omega_1$ . Then  $\Omega_2$  is subjected to the reaction  $-\mathbf{r}$ , acting from  $\Omega_1$ . Let  $\mathbf{n}$  denote the normal unit vector at point M to the bodies, directed towards  $\Omega_1$ , and  $\mathbf{T}(t_1, t_2)$  denotes the orthogonal plane to  $\mathbf{n}$  in  $\mathcal{R}^3$ . Any element  $\dot{\mathbf{u}}$  and  $\mathbf{r}$  may uniquely be decomposed in the form:

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_t + \dot{u}_n \mathbf{n}, \quad \dot{\mathbf{u}}_t \in \mathbf{T}, \quad \dot{u}_n \in \mathcal{R} \quad (1)$$

$$\mathbf{r} = \mathbf{r}_t + r_n \mathbf{n}, \quad \mathbf{r}_t \in \mathbf{T}, \quad r_n \in \mathcal{R} \quad (2)$$

Classically, a unilateral contact law is characterized by a geometric condition of non-penetration, a static condition of no-adhesion and a mechanical complementarity condition. These three conditions are the so-called Signorini conditions written in terms of the signed contact distance  $x_n$  and the normal contact force  $r_n$ :

$$x_n \geq 0; \quad r_n \geq 0 \quad \text{and} \quad r_n x_n = 0 \quad (3)$$

where  $x_n$  denotes the magnitude of the gap between the contact node and the target surface and is a violation of the contact compatibility:

$$x_n = g + u_n \quad (4)$$

where  $g$  denotes the initial gap between the solids in contact.

In the case of dynamic analysis such as impact problems, the Signorini's condition is described in terms of velocity in conjunction with the sliding rule. Eq.(3) is equivalent to

$$\dot{u}_n \geq 0; \quad r_n \geq 0 \quad \text{and} \quad r_n \dot{u}_n = 0 \quad (5)$$

Let  $K_\mu$  denote Coulomb's convex cone:

$$K_\mu = \{\mathbf{r} \in \mathfrak{R}^3 \text{ such that } |\mathbf{r}_t| \leq \mu r_n\} \quad (6)$$

$K_\mu$  is a closed convex set. The complete contact law is a complex non-smooth dissipative law including three statuses: no contact, contact with sticking and contact with sliding. The resulting analytical transcripts yield two overlapped "if...then...else" statements:

$$\begin{cases} \text{if } r_n = 0, \text{ then } \dot{u}_n \geq 0, & ! \text{ no contact} \\ \text{else if } \mathbf{r} \in K_\mu \text{ then } \dot{\mathbf{u}} = 0, & ! \text{ sticking} \\ \text{else } (r_n > 0 \text{ and } \mathbf{r} \in \partial K_\mu), \left\{ \dot{u}_n \geq 0 \text{ and } \exists \lambda \geq 0 \text{ such that } \dot{\mathbf{u}}_t = -\lambda \frac{\mathbf{r}_t}{|\mathbf{r}_t|} \right\} & ! \text{ sliding} \end{cases} \quad (7)$$

De Saxcé & Feng [11] have shown that the contact law (7) is equivalent to

$$-(\dot{\mathbf{u}}_t + (\dot{u}_n + \mu |\dot{\mathbf{u}}_t|) \mathbf{n}) \in \partial \bigcup_{K_\mu} (\mathbf{r}) \quad (8)$$

where  $\bigcup_{K_\mu} (\mathbf{r})$  denotes the so-called indicatory function of the closed convex set  $K_\mu$ :

$$\bigcup_{K_\mu} (\mathbf{r}) = \begin{cases} 0, & \text{if } \mathbf{r} \in K_\mu \\ +\infty & \text{otherwise} \end{cases} \quad (9)$$

The following contact bi-potential is obtained:

$$b_c(-\dot{\mathbf{u}}, \mathbf{r}) = \bigcup_{\mathfrak{R}_-} (-\dot{u}_n) + \bigcup_{K_\mu} (\mathbf{r}) + \mu r_n |\dot{\mathbf{u}}_t| \quad (10)$$

where  $\mathfrak{R}_-$  is the set of the negative and null real numbers.

### Local algorithm

In order to avoid nondifferentiable potentials that occur in nonlinear mechanics, such as in contact problems, it is convenient to use the Augmented Lagrangian Method [8-12]. Let  $\rho > 0$  be chosen in a suitable range to ensure numerical convergence. For the contact bi-potential  $b_c$ , given by (10), provided that  $\dot{u}_n \geq 0$  and  $\mathbf{r} \in K_\mu$ , we have

$$\forall \mathbf{r}' \in K_\mu \quad \rho \mu (r_n' - r_n) |\dot{\mathbf{u}}_t| + [\mathbf{r} - (\mathbf{r} - \rho \dot{\mathbf{u}})](\mathbf{r}' - \mathbf{r}) \geq 0 \quad (11)$$

Taking account of the decomposition (1), the following inequality has to be satisfied:

$$\forall \mathbf{r}' \in K_\mu \quad (\mathbf{r} - \boldsymbol{\tau}) \cdot (\mathbf{r}' - \mathbf{r}) \geq 0 \quad (12)$$

where the modified augmented surface traction  $\boldsymbol{\tau}$  is defined by:

$$\boldsymbol{\tau} = \mathbf{r} - \rho (\dot{\mathbf{u}}_t + (\dot{u}_n + \mu |\dot{\mathbf{u}}_t|) \mathbf{n}) \quad (13)$$

The inequality (12) means that  $\mathbf{r}$  is the projection of  $\boldsymbol{\tau}$  onto the closed convex Coulomb's cone:

$$\mathbf{r} = \text{proj}(\boldsymbol{\tau}, K_\mu) \quad (14)$$

For the numerical solution of the implicit equation (14), Uzawa's algorithm can be used, which leads to an iterative process involving one predictor-corrector step:

$$\text{Predictor: } \boldsymbol{\tau}^{i+1} = \mathbf{r}^i - \rho^i (\dot{\mathbf{u}}_t^i + (\dot{u}_n^i + \mu |\dot{\mathbf{u}}_t^i|) \mathbf{n}) \quad (15)$$

$$\text{Corrector: } \mathbf{r}^{i+1} = \text{proj}(\boldsymbol{\tau}^{i+1}, K_\mu) \quad (16)$$

It is worth noting that, in this algorithm, the unilateral contact and the friction are coupled via the bi-potential. Another gist of the bi-potential method is that the corrector can be analytically found with respect to the three possible contact statuses:  $\boldsymbol{\tau} \in K_\mu$  (contact with sticking),  $\boldsymbol{\tau} \in K_\mu^*$  (no contact) and  $\boldsymbol{\tau} \in \mathfrak{R}^3 - (K_\mu \cup K_\mu^*)$  (contact with sliding).  $K_\mu^*$  is the polar cone of  $K_\mu$ . This corrector step is explicitly given as follows:

$$\left\{ \begin{array}{lll} \text{if} & \mu |\boldsymbol{\tau}_t^{i+1}| < -\tau_n^{i+1} & \text{then } \mathbf{r}^{i+1} = 0 \quad ! \text{ no contact} \\ \text{else if} & |\boldsymbol{\tau}_t^{i+1}| \leq \mu \tau_n^{i+1} & \text{then } \mathbf{r}^{i+1} = \boldsymbol{\tau}^{i+1} \quad ! \text{ sticking} \\ \text{else} & \mathbf{r}^{i+1} = \boldsymbol{\tau}^{i+1} - \frac{(|\boldsymbol{\tau}_t^{i+1}| - \mu \tau_n^{i+1})}{(1+\mu^2)} \left( \frac{\boldsymbol{\tau}_t^{i+1}}{|\boldsymbol{\tau}_t^{i+1}|} + \mu \mathbf{n} \right) & ! \text{ sliding} \end{array} \right. \quad (17)$$

It is important to emphasize on the fact that this explicit formula is valid for both 2D and 3D contact problems with Coulomb's friction and allows us to obtain very stable and accurate results.

## 2. First order integration algorithm

Generally, non-linear dynamic mechanical behaviors of solid media with contact are governed by an equilibrium equation (after finite element discretization):

$$\mathbf{M} \ddot{\mathbf{u}} = \mathbf{F} + \mathbf{R}_c \quad (18)$$

where  $\mathbf{F} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{int}} - \mathbf{C} \dot{\mathbf{u}}$ .

The vectors  $\mathbf{F}_{\text{int}}$ ,  $\mathbf{F}_{\text{ext}}$  and  $\mathbf{R}_c$  denote, respectively, the internal, external and contact forces.  $\mathbf{M}$  is the mass matrix and  $\mathbf{C}$  the damping matrix.  $\dot{\mathbf{u}}$  is the velocity vector and  $\ddot{\mathbf{u}}$  the acceleration vector. It is noted that the stiffness effect is taken into account by the internal forces vector  $\mathbf{F}_{\text{int}}$ . The most common method for integrating the dynamics equation (18) is the Newmark method which is based on a second order algorithm. However, in impact problems, higher order approximation does not necessarily mean better accuracy, and may even be superfluous. At the moment of sudden change of contact conditions (impact, release of contact), the velocity and acceleration are not continuous, and excessive regularity constraints may lead to serious errors. For this reason, Jean [19] has proposed a first order algorithm which is used in this work. Eq.(18) can be transformed into

$$\mathbf{M} d\dot{\mathbf{u}} = \mathbf{F} dt + \mathbf{R}_c dt \quad (19)$$

We can now integrate (19) between consecutive time configuration  $t$  and  $t+\Delta t$ , using the following approximations:

$$\int_t^{t+\Delta t} \mathbf{M} d\dot{\mathbf{u}} = \mathbf{M} (\dot{\mathbf{u}}^{t+\Delta t} - \dot{\mathbf{u}}^t) \quad (20)$$

$$\int_t^{t+\Delta t} \mathbf{F} dt = \Delta t \left( (1-\xi) \mathbf{F}^t + \xi \mathbf{F}^{t+\Delta t} \right) \quad (21)$$

$$\int_t^{t+\Delta t} \mathbf{R}_c dt = \Delta t \mathbf{R}_c^{t+\Delta t} \quad (22)$$

$$\mathbf{u}^{t+\Delta t} - \mathbf{u}^t = \Delta t \left( (1-\theta) \dot{\mathbf{u}}^t + \theta \dot{\mathbf{u}}^{t+\Delta t} \right) \quad (23)$$

where  $0 \leq \xi \leq 1$ ;  $0 \leq \theta \leq 1$ . In the iterative solution procedure, all the values at time  $t+\Delta t$  are replaced by the values of the current iteration  $i+1$ ; for example,  $\mathbf{F}^{t+\Delta t} = \mathbf{F}^{i+1}$ . A standard approximation of  $\mathbf{F}^{i+1}$  gives

$$\mathbf{F}^{i+1} = \mathbf{F}_{\text{int}}^i + \frac{\partial \mathbf{F}}{\partial \mathbf{u}} (\mathbf{u}^{i+1} - \mathbf{u}^i) + \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{u}}} (\dot{\mathbf{u}}^{i+1} - \dot{\mathbf{u}}^i) = \mathbf{F}_{\text{int}}^i - \mathbf{K}^i \Delta \mathbf{u} - \mathbf{C}^i \Delta \dot{\mathbf{u}} \quad (24)$$

Finally, we obtain the recursive form of (19) in terms of displacements

$$\begin{cases} \bar{\mathbf{K}}^i \Delta \mathbf{u} = \bar{\mathbf{F}}^i + \bar{\mathbf{F}}_{\text{acc}}^i + \mathbf{R}_c^{i+1} \\ \mathbf{u}^{i+1} = \mathbf{u}^i + \Delta \mathbf{u} \end{cases} \quad (25)$$

where the so-called effective terms are given by

$$\bar{\mathbf{K}}^i = \xi \mathbf{K}^i + \frac{\xi}{\theta \Delta t} \mathbf{C}^i + \frac{1}{\theta \Delta t^2} \mathbf{M}^i \quad (26)$$

$$\bar{\mathbf{F}}_{\text{acc}}^i = -\frac{1}{\theta \Delta t^2} \mathbf{M}^i \left\{ \mathbf{u}^i - \mathbf{u}^t - \Delta t \dot{\mathbf{u}}^t \right\} \quad (27)$$

$$\bar{\mathbf{F}}^i = (1 - \xi) \left( \mathbf{F}_{\text{int}}^t + \mathbf{F}_{\text{ext}}^t \right) + \xi \left( \mathbf{F}_{\text{int}}^i + \mathbf{F}_{\text{ext}}^{t+\Delta t} \right) \quad (28)$$

At the end of each time step, the velocity is updated by

$$\dot{\mathbf{u}}^{t+\Delta t} = \left( 1 - \frac{1}{\theta} \right) \dot{\mathbf{u}}^t + \frac{1}{\theta \Delta t} \left( \mathbf{u}^{t+\Delta t} - \mathbf{u}^t \right) \quad (29)$$

The equation (25) is strongly non-linear, because of finite strains and large displacements of solid, for instance in large deformation problems. Besides, the constitutive law of contact with friction is usually represented by inequalities and the contact potential is even nondifferentiable. Instead of solving this equation in consideration of all nonlinearities at the same time, Feng [20] has proposed a solution strategy which consists in separating the nonlinearities in order to overcome the complexity of calculation and to improve the numerical stabilities. As  $\Delta \mathbf{u}$  and  $\mathbf{R}_c$  are both unknown, Eq.(25) cannot be directly solved. First, the vector  $\mathbf{R}_c$  is determined by the bi-potential method (Eqs.15-17) in a reduced system, which only concerns contact nodes. Then, the vector  $\Delta \mathbf{u}$  can be computed in the whole structure, using contact reactions as external loading. It is very important to note that, as opposed to the penalty method or Lagrange multiplier method, the bi-potential method neither changes the global stiffness matrix, nor increases the degrees of freedom. One consequence of this interesting property is that it is easy to implement contact and friction problems in an existing general-purpose finite element code by this method. In addition, the solution procedure is more stable because of the separation of nonlinearities and improved numerical algorithms for calculation of contact reactions.

### 3. Energy computation

Because of large displacements and rotations in the case of dynamic multibody contact problems, Green-Lagrangian strain is adopted for the nonlinear relationships between strains and displacements. We note  $\Phi$  the deformation gradient tensor. The Green-Lagrangian strain tensor  $\mathbf{E}$  is defined by

$$\mathbf{E} = \frac{1}{2} \left( \Phi^T \Phi - \mathbf{I} \right) \quad (30)$$

In the case of elastic laws, there exists an elastic potential function  $W$  (or strain energy density function) which is a scalar function of the strain tensor. In the particular case of isotropic Saint-Venant-Kirchhoff model,  $W$  is defined by

$$W = \frac{1}{2} \mathbf{E} : \mathbf{D} \mathbf{E} \quad (31)$$

where  $\mathbf{D}$  denotes the usual material secant tangent.

The total elastic strain energy of the contact bodies (discretized by  $n_{\text{el}}$  elements) is then written by

$$E_e = \sum_{\alpha=1}^{n_{el}} \int_{\Omega_\alpha} W_\alpha d\Omega \quad (32)$$

The total kinetic energy can be calculated at the global level by

$$E_k = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} \quad (33)$$

Finally, the total energy of the system of solids is

$$E_t = E_e + E_k \quad (34)$$

The case of interest for the analysis presented below corresponds to the homogeneous Neumann problem, characterized by no imposed boundary displacements and no external loading. In addition, if frictionless contact is considered, the total energy should be conserved. The proposed algorithm preserves this fundamental energy conservation property as shown in numerical examples.

#### 4. Numerical results

The algorithms presented above have been implemented and tested in the finite element code FER/Contact [21]. Many application examples, in static or quasi-static cases, have been carried out using the present method [20-23].

The first example of dynamics analysis will be presented to show the validity and efficiency of the model developed. The problem concerns the oblique impact of a 2D elastic plate onto a rigid surface with rebounding. This example has been proposed and studied by Kwak *et al.* [24, 25] using Linear Complementarity Problem (LCP) formulation. The geometric configuration and successive deformed meshes are displayed in Figure 1. For comparison purpose, we have used the same mesh as in [25].

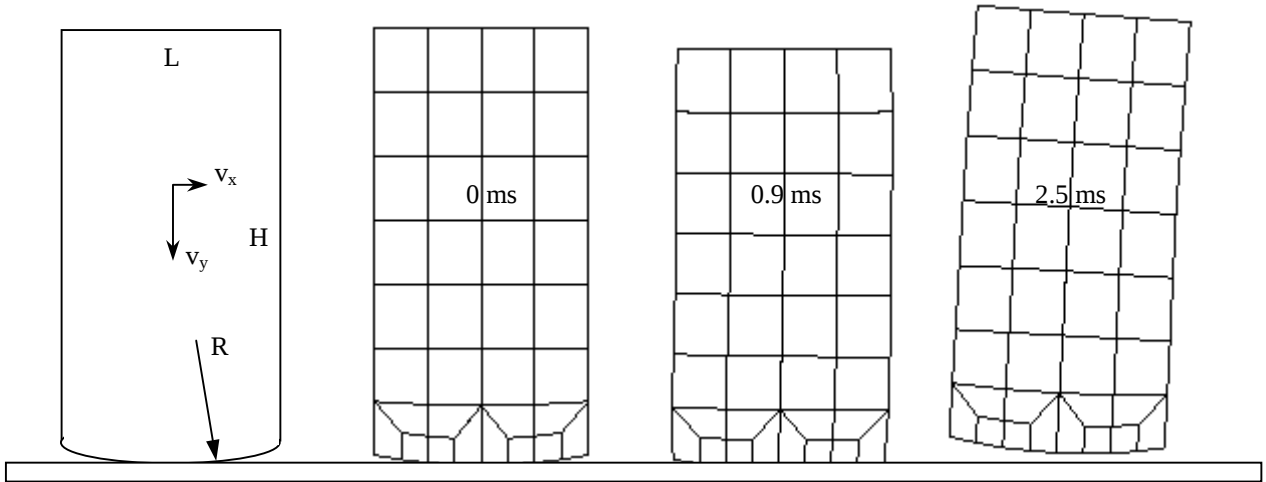


Fig. 1. Oblique impact of an elastic plate: geometry and deformed shapes vs time

The characteristics of this example are: Young's modulus  $E = 10^7$  Pa, Poisson's ratio  $\nu = 0.25$ , mass density  $\rho = 1000$  kg/m<sup>3</sup>, Friction coefficient  $\mu = 0.1$ , Initial velocity:  $v_x = 3$  m/s,  $v_y = -5$  m/s. The geometric sizes are:  $L = 0.04$  m,  $H = 0.08$  m, radius  $R = 0.101$  m, thickness  $e = 0.01$  m. The integration time is  $3 \cdot 10^{-3}$  s and the solution parameters are:  $\Delta t = 10^{-5}$  s,  $\theta = \xi = 0.5$ . The plate is modeled by 54 nodes and 37 linear quadrilateral plane stress elements (Figure 3). The performance of the present approach in terms of CPU time, as compared to Kwak's solutions, is reported in Table 1, which shows the efficiency of the proposed method.

Table 1. Comparison of CPU time

| Method          | Computer             | CPU time (s) |
|-----------------|----------------------|--------------|
| Ko & Kwak [24]  | CRAY 2S/ 4-128       | 19 000       |
| Kim & Kwak [25] | HP 720               | 430          |
| present         | PC Pentium 4/2.8 GHz | 7            |

The second example simulates the impact of two cylinders inside rigid walls (proposed by Armero & Petocz [15]). In doing so, we wish to further explore the performance of the present method in a general situation with complicated contact sequences. The problem is displayed in Figure 2. The cylinders have a diameter of 2. The Saint-Venant-Kirchhoff material model is assumed for both cylinders with material constants:  $E = 2700$ ,  $\nu = 0.33$ , and mass density  $\rho = 1$ . The left cylinder is given an initial velocity of  $v_x = 1$ ,  $v_y = -2$ , hitting the bottom rigid wall and afterwards the right cylinder as depicted in Figure 2. The total simulation time is 15 s and the time step is  $\Delta t = 10^{-3}$  s. Figure 3 shows the plots of the kinetic energy  $E_k$ , the elastic strain energy  $E_e$  and the total energy  $E_t$ . We can observe clearly that the total energy is perfectly conserved in the case of frictionless contact. However, in the case of frictional contact ( $\mu = 0.2$ ), the total energy decreases at each shock (Figure 4). So the energy is dissipated by frictional effects as expected.

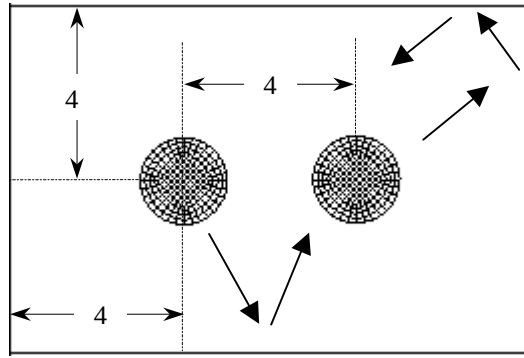


Fig. 2. Impact of two cylinders inside rigid walls

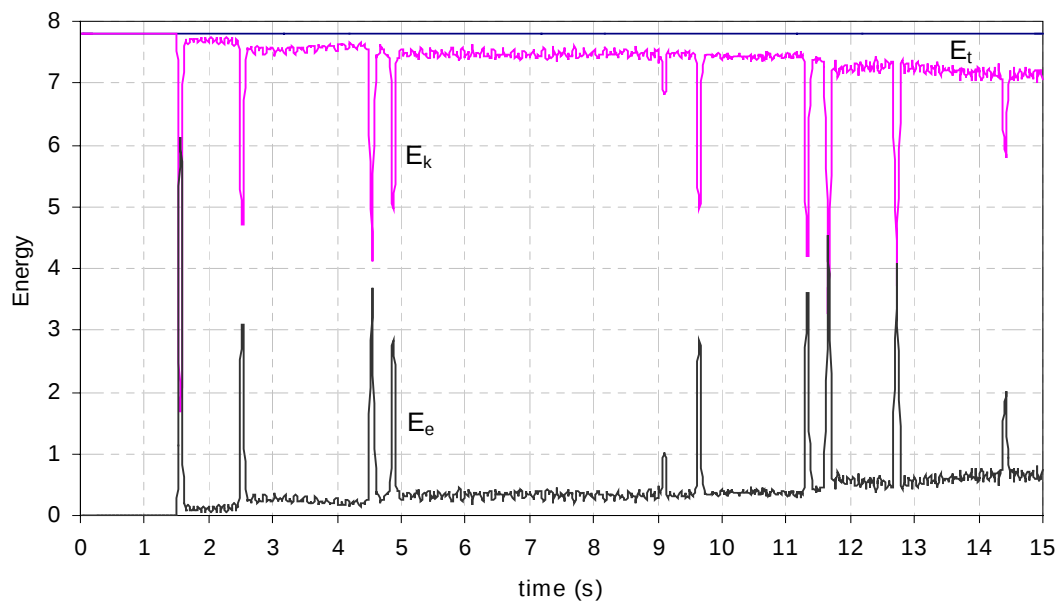


Fig. 3. Energy evolution without friction ( $\mu = 0.0$ )

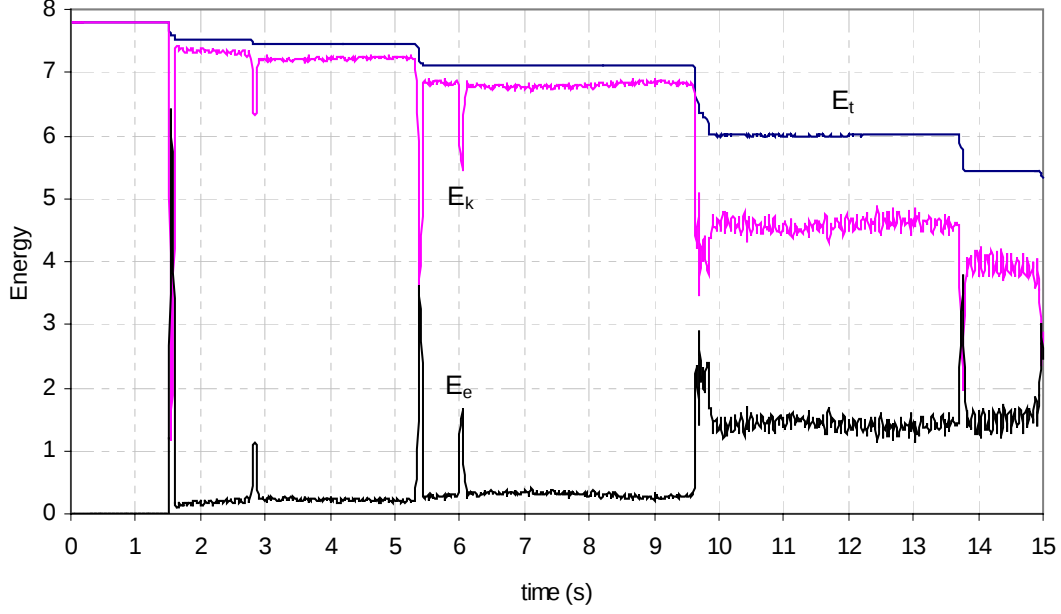


Fig. 4. Energy evolution with friction ( $\mu = 0.2$ )

It is interesting to note from Figure 3 that the left cylinder hits another one at  $t \approx 2.45$  s. On the other hand,  $t \approx 2.8$  s in the case of frictional contact as indicated in Figure 4. This fact can be explained as follows: because of friction forces, the rebounding direction is changed such that the running distance of the left cylinder from the bottom wall to the right cylinder becomes longer. Thus, it takes more time to hit each other. Figure 5 shows the distribution of the shear stress of the left cylinder when it hits the bottom wall. Without friction, the distribution is symmetric, but this is not true with friction. Once again, the frictional effects are apparently demonstrated.

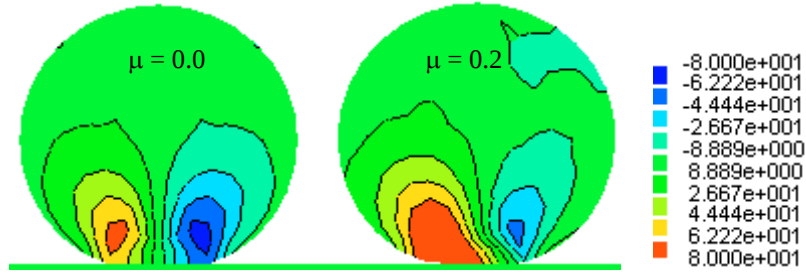


Fig. 5. Distribution of shear stress at  $t = 1.56$  s

## 5. Conclusion

In this paper, we have presented the recent development of the bi-potential method applied to dynamic analysis in contact mechanics. The numerical algorithms are described. The above results demonstrate that the proposed algorithms, for the local analysis of frictional contact problems and for the global resolution of dynamics equation using the first order algorithm can provide good results in terms of numerical stability and precision. The proposed algorithms not only preserve the energy conservation property of frictionless contact of solids but also take into account the physical energy dissipation by frictional effects.

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