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# A PRECONDITIONER FOR THE SCHUR COMPLEMENT DOMAIN DECOMPOSITION METHOD


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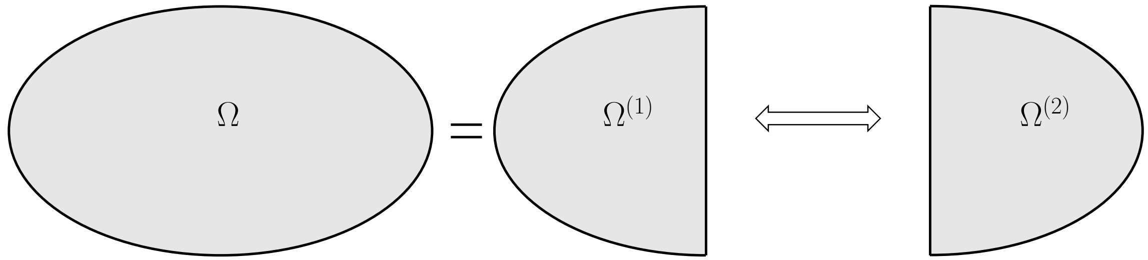
## OUTLINE

- ❐ THE SCHUR COMPLEMENT DOMAIN DECOMPOSITION METHOD.
- ❐ PRECONDITIONER : CONTINUOUS FIELD AT THE CROSS-POINTS.
- ❐ COMPUTATIONAL RESULTS : STRUCTURAL ANALYSIS PROBLEMS.

FE ON $\Omega$ + NON-OVERLAPPING DECOMPOSITION




$$u^{(1)} = u^{(2)}$$
$$[\sigma^{(1)}\, n^{(1)} + \sigma^{(2)}\, n^{(2)}] = 0$$

- ❒ CONDENSATION : ELIMINATION OF INTERIOR DOF OF LOCAL SUBDOMAIN MATRICES
- ❒ ITERATIVE SOLUTION OF THE REDUCED PROBLEM ON INTERFACE
- ❒ NUMERICAL AND PARALLEL SCALABILITY

PRECONDITIONER

(NEUMANN PROBLEM IN EACH SUBDOMAIN + AVERAGING)

+

COARSE PROBLEM

(RIGID BODY MODES OR CORNER MODES)

- NOTATIONS

$$K^s = \begin{bmatrix} K^s_{ii} & K^s_{ib} \\ {K^s_{ib}}^T & K^s_{bb} \end{bmatrix}, \quad u^s = \begin{Bmatrix} u^s_i \\ u^s_b \end{Bmatrix}, f^s = \begin{Bmatrix} f^s_i \\ f^s_b \end{Bmatrix}, \quad \text{AND} \quad u^s_b = {N^s_b}^T u_b.$$

- GLOBAL LINEAR SYSTEM

$$\begin{bmatrix} K^1_{ii} & \cdots & 0 & K^1_{ib}{N^1_b}^T \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & K^{n_s}_{ii} & K^{n_s}_{ib}{N^{n_s}_b}^T \\ N^1_b {K^1_{ib}}^T & \cdots & N^{n_s}_b {K^{n_s}_{ib}}^T & \sum_{s=1}^{n_s} N^s_b K^s_{bb} {N^s_b}^T \end{bmatrix} \begin{Bmatrix} u^1_i \\ \vdots \\ u^{n_s}_i \\ u_b \end{Bmatrix} = \begin{Bmatrix} f^1_i \\ \vdots \\ f^{n_s}_i \\ \sum_{s=1}^{n_s} N^s_b \; f^s_b \end{Bmatrix}.$$

- INTERFACE PROBLEM

$$\left( \sum_{s=1}^{n_s} N^s_b \left( K^s_{bb} - {K^s_{ib}}^T {K^s_{ii}}^{-1} K^s_{ib} \right) {N^s_b}^T \right) u_b = \sum_{s=1}^{n_s} N^s_b \left( f^s_b - {K^s_{ib}}^T {K^s_{ii}}^{-1} f^s_i \right).$$

- WITHOUT COARSE GRID

$$M = \sum_{s=1}^{n_s} N_b^s \; D^s \; {S^s}^{-1} \; D^s \; {N_b^s}^T,$$

WITH $$\sum_{s=1}^{n_s} N_b^s \; D^s \; {N_b^s}^T = I_\Gamma.$$

- WITH COARSE GRID

$Z^s$, RIGID BODY MODES OR CORNER MODES

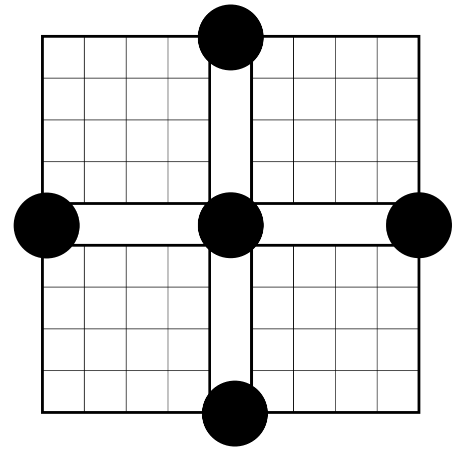
$$G = [N_b^1 D^1 Z^1, ..., N_b^{n_s} D^{n_s} Z^{n_s}]$$

$$M = \left(I - G \; [G^T \; S \; G]^{-1} \; G^T S\right) \sum_{s=1}^{n_s} N_b^s \; D^s \; {\tilde{S}^s}^{-1} \; D^s \; {N_b^s}^T.$$

EFFICIENCY, BUT CONSTRUCTION OF THE COARSE PROBLEM IS COSTLY

CORNERS OR CROSS-POINTS :

- ➪ CROSS-POINTS, POINTS BELONGING TO MORE THAN TWO SUBDOMAINS
- ➪ POINTS LOCATED AT THE BEGINNING AND END OF EACH EDGE OF EACH SUDOMAIN



PRECONDITIONED CONJUGATE GRADIENT - MECHANICAL INTERPRETATION

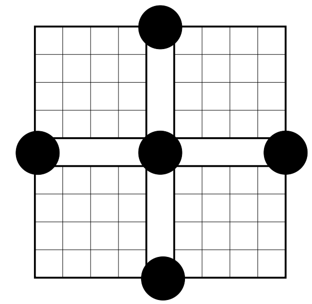
GRADIENT = FORCE

PRECONDITIONED GRADIENT = DISPLACEMENT

$\Downarrow$

IMPOSE CONTINUOUS DISPLACEMENT AT THE CROSS-POINTS

$$\{u\} = \left\{ \begin{array}{c} u_r \\ u_c \end{array} \right\} = \left\{ \begin{array}{c} u_r^1 \\ \vdots \\ u_r^{n_s} \\ u_c \end{array} \right\}, \quad u_c^s = N_c^{s^T} \ u_c.$$



$$\left[ \begin{array}{cccc} K_{rr}^1 & \cdots & 0 & K_{rc}^1 N_c^{1^T} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & K_{rr}^{n_s} & K_{rc}^{n_s} N_c^{n_s^T} \\ N_c^1 K_{rc}^{1^T} & \cdots & N_c^{n_s} K_{rc}^{n_s^T} & \sum_{s=1}^{n_s} N_c^s K_{cc}^s N_c^{s^T} \end{array} \right] \left\{ \begin{array}{c} u_r^1 \\ \vdots \\ u_r^{n_s} \\ u_c \end{array} \right\} = \left\{ \begin{array}{c} f_r^1 \\ \vdots \\ f_r^{n_s} \\ \sum_{s=1}^{n_s} N_c^s \ f_c^s \end{array} \right\}.$$

$$u_c = \left( \sum_{s=1}^{n_s} N_c^s \left( K_{cc}^s - K_{rc}^{s^T} K_{rr}^{s^{-1}} K_{rc}^s \right) N_c^{s^T} \right)^{-1} \sum_{s=1}^{n_s} N_c^s \left( f_c^s - K_{rc}^{s^T} K_{rr}^{s^{-1}} f_r^s \right),$$

$$u_r^s = K_{rr}^{s^{-1}} \left( f_r^s - K_{rc}^s \ N_c^{s^T} \ u_c \right).$$

CONNECTION WITH THE INTERFACE

$$u_r^s = R_r^s\, u_b^s, \quad u_c^s = R_c^s\, u_b^s.$$

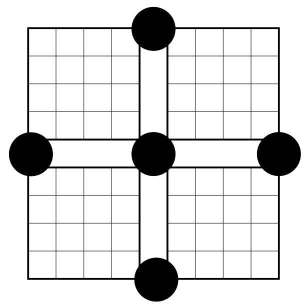
$$M = \sum_{s=1}^{n_s} N_b^s D^s \left[ R_r^{s^T} K_{rr}^{s^{-1}} R_r^s \right] D^s N_b^{s^T} +$$

$$+ \sum_{s=1}^{n_s} N_b^s D^s \left[ \left( N_c^s (R_c^s - K_{rc}^{s^T} K_{rr}^{s^{-1}} R_r^s) \right)^T S_c^{-1} \sum_{s=1}^{n_s} \left( N_c^s\, (R_c^s - K_{rc}^{s^T} K_{rr}^{s^{-1}} R_r^s) \right) \right] D^s N_b^{s^T}$$

ADD ARTIFICIAL CROSS-POINTS TO GET NON-LINEAR (NON-PLANAR IN 3D) CROSS-POINTS !!!

NEW COARSE PROBLEM

$$S_c = \sum_{s=1}^{n_s} N_c^s \left[ K_{cc}^s - K_{rc}^{s^T} K_{rr}^{s^{-1}} K_{rc}^s \right] N_c^{s^T} = \sum_{s=1}^{n_s} N_c^s\; S_c^s\; N_c^{s^T}$$



5 CROSS-POINTS

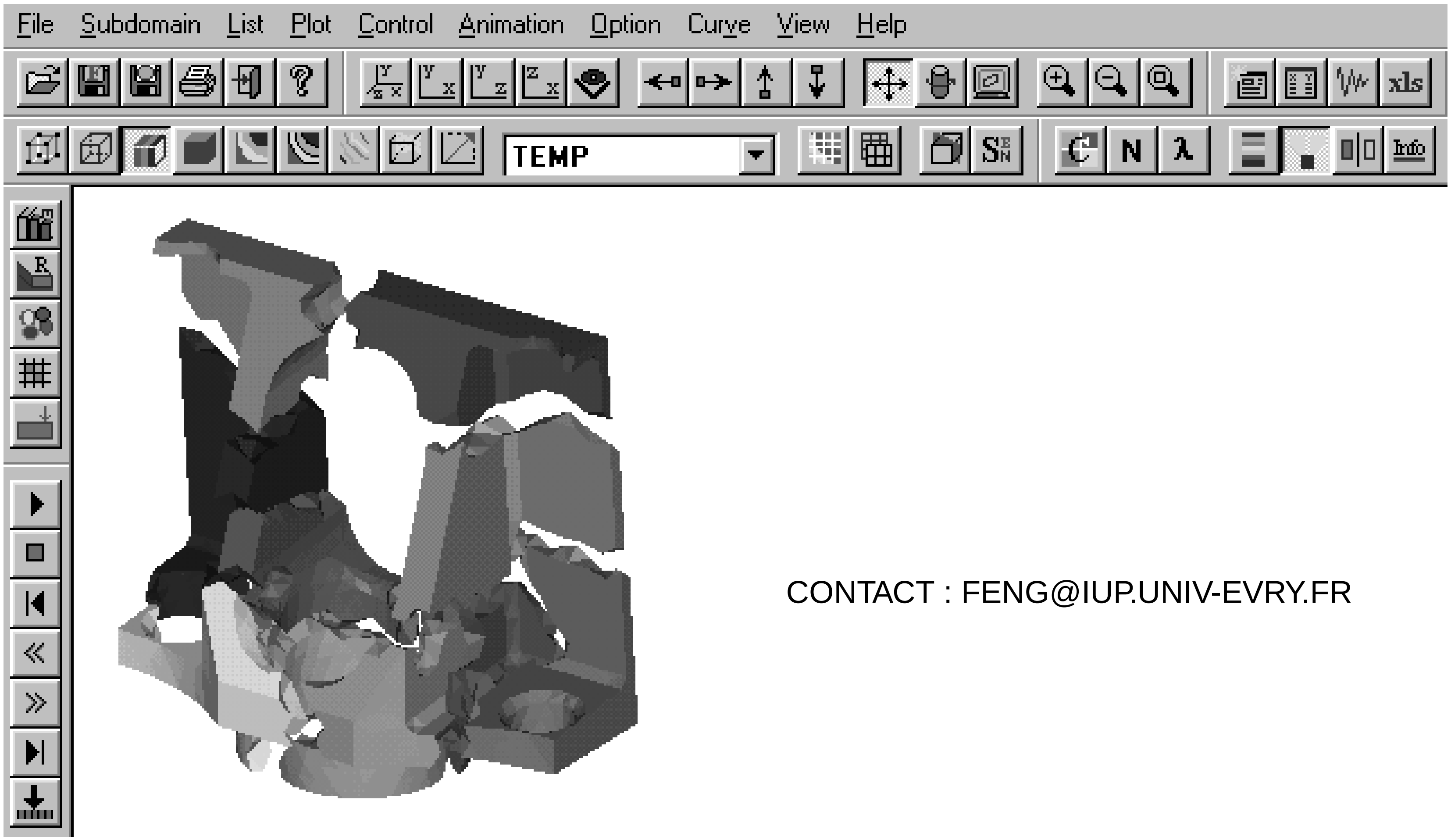
BUT 12 CORNER MODES!!!

- ❒ NEUMANN-NEUMANN PRECONDITIONER + CONTINUITY (COARSE PROBLEM = 5)
  - IN EACH SUBDOMAIN : REDUCED PROBLEM ON THE CROSS-POINTS $S_c^s$ (SCHUR COMPLEMENT MATRIX)
  - ASSEMBLING OF $S_c = \sum_{s=1}^{n_s} N_c^s S_c^s {N_c^s}^T$ AND FACTORISATION OF THE COARSE PROBLEM

- ❒ GENERALIZED NEUMANN-NEUMANN PRECONDITIONER (COARSE PROBLEM = 12)
  - IN EACH SUBDOMAIN : COMPUTATION OF THE CORNER MODES $Z^s$
  - EXCHANGE OF THE CORNER MODES WITH THE NEIGHBORING SUBDOMAINS
  - COMPUTATION OF $SG$
  - ASSEMBLING OF $G^T SG$ AND FACTORISATION OF THE COARSE PROBLEM

- DECOMPOSITION INTO SUBDOMAINS BY METIS 4.0.1 (FER/SUBDOMAIN)
- CODE WRITTEN IN C++ AND FORTRAN 77
- COMMUNICATIONS : MPI
- ONE SUBDOMAIN BY NODE
- FINITE ELEMENT (MODULEF,...)
- SKYLINE SOLVERS IN THE SUBDOMAINS
- COARSE PROBLEM IS ASSEMBLED AND SOLVED BY DIRECT SOLVER
- OPTIMISATION : BLAS, LAPACK,...
- STOPPING CRITERION : $\| K\, u - f \| \,/\, \| f \| \;\; \leq \; 10^{-6}$
- SGI ORIGIN 2000 (64 PROCS.),
  PÔLE PARALLÉLISME ÎLE DE FRANCE SUD, ENS CACHAN

OBJECT-ORIENTED PROGRAM WRITTEN IN C++ AND OPEN-GL (WINDOWS, IRIX)

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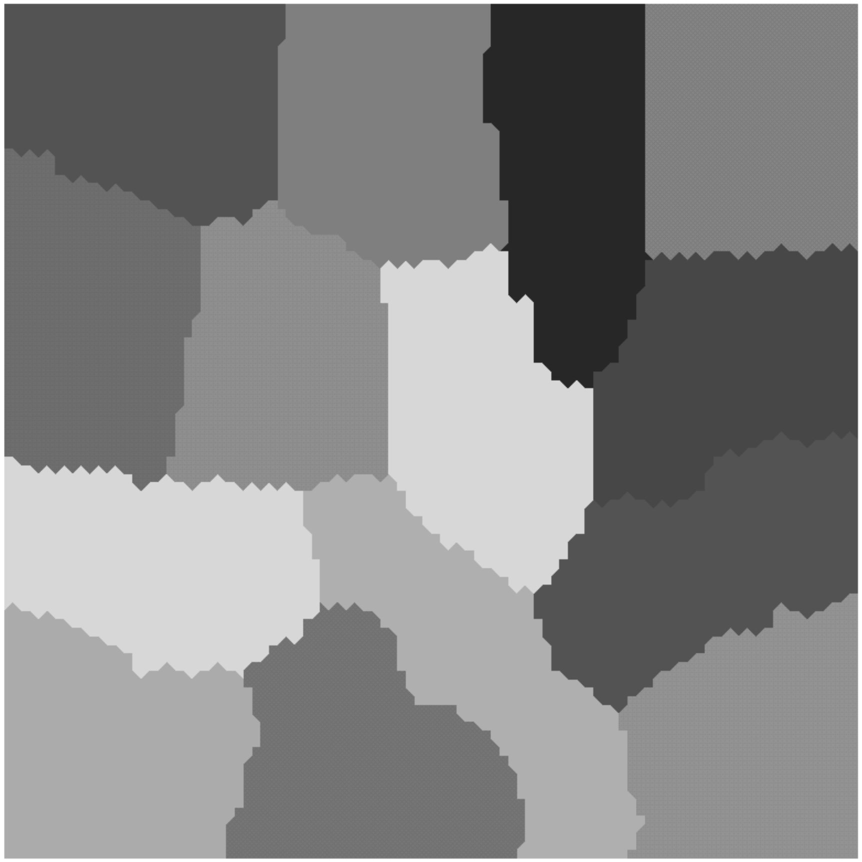
- MESH PARTITIONING (METIS,...)
- RENUMBERING
- CROSS-POINTS DETECTION
- DESCRIPTION OF THE INTERFACE

## 2D ELASTICITY : 1 × 1 m

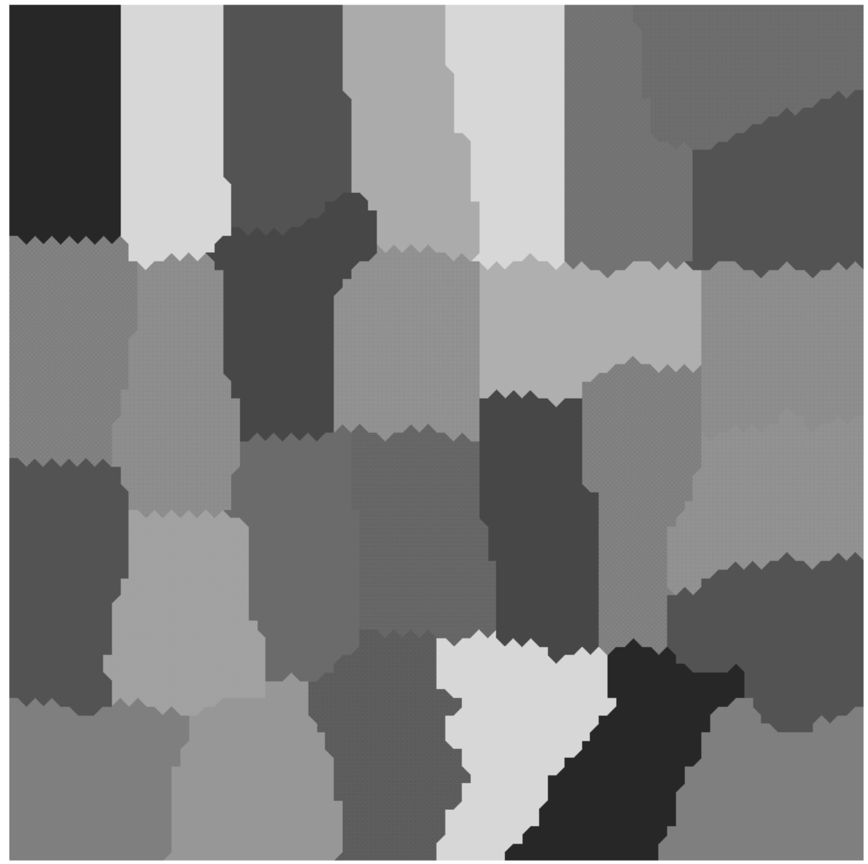
3-NODE ELEMENTS (101 × 101 NODES, 20 402 DOF)

14 SUBDOMAINS (INTERFACE = 1 202 DOF) 28 SUBDOMAINS (INTERFACE = 1 878 DOF)

3-NODE ELEMENTS (101 $\times$ 101 NODES, 20 402 DOF)



14 SUBDOMAINS (INTERFACE = 1 202 DOF)



28 SUBDOMAINS (INTERFACE = 1 878 DOF)

## 2D ELASTICITY

| | PRIMAL | | | DUAL | | | |
|---|---|---|---|---|---|---|---|
| | | | | FETI-DP | | FETI-1 | |
| $n_s$ | GNN (RBM) | GNN (CM) | NN+C | DIRICHLET | LUMPED | DIRICHLET | LUMPED |
| 14 | 24 iter. | 14 iter. | 20 iter. | 22 iter. | 60 iter. | 24 iter. | 49 iter. |
| 28 | 26 iter. | 14 iter. | 23 iter. | 24 iter. | 66 iter. | 26 iter. | 52 iter. |

SIZE OF THE COARSE PROBLEM

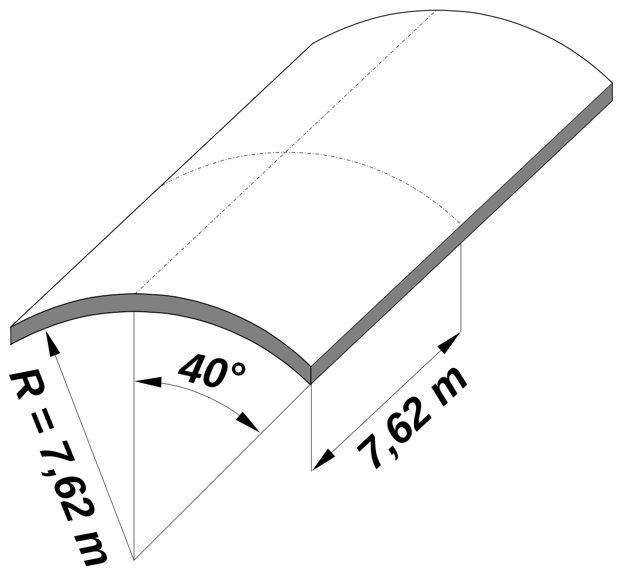
| | PRIMAL | | | DUAL | | | |
|---|---|---|---|---|---|---|---|
| | | | | FETI-DP | | FETI-1 | |
| $n_s$ | GNN (RBM) | GNN (CM) | NN+C | DIRICHLET | LUMPED | DIRICHLET | LUMPED |
| 14 | 30 | 132 | 52 | | | 30 | |
| 28 | 72 | 286 | 108 | | | 72 | |

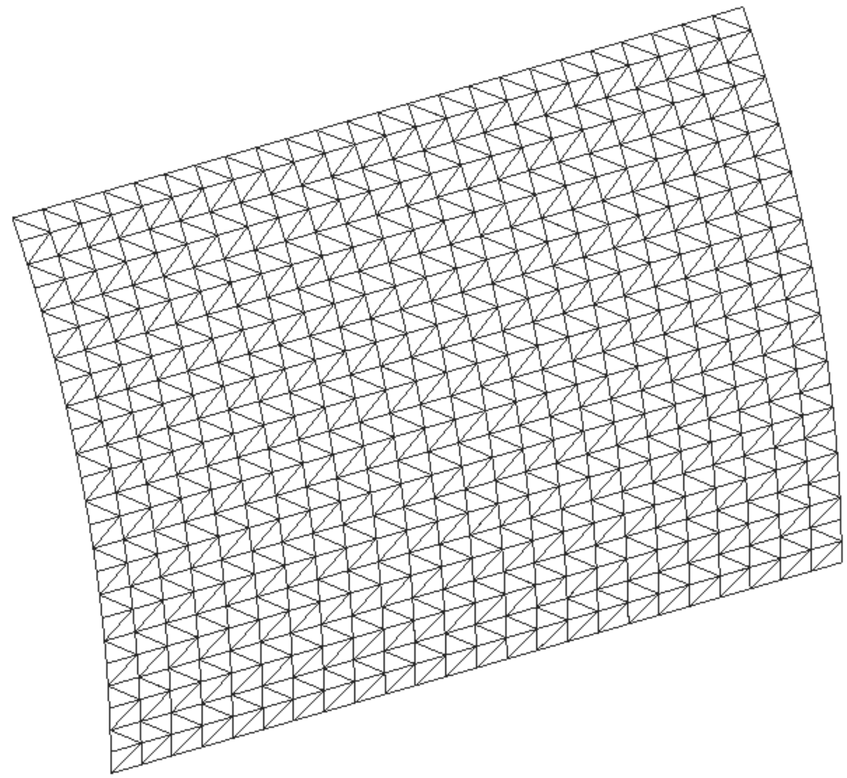
GNN : GENERALIZED NEUMANN-NEUMANN PRECONDITIONER

(RBM : RIGID BODY MODES, CM : CORNER MODES)

NN+C : NEUMANN-NEUMANN PRECONDITIONER + CONTINUITY AT THE CROSS-POINTS

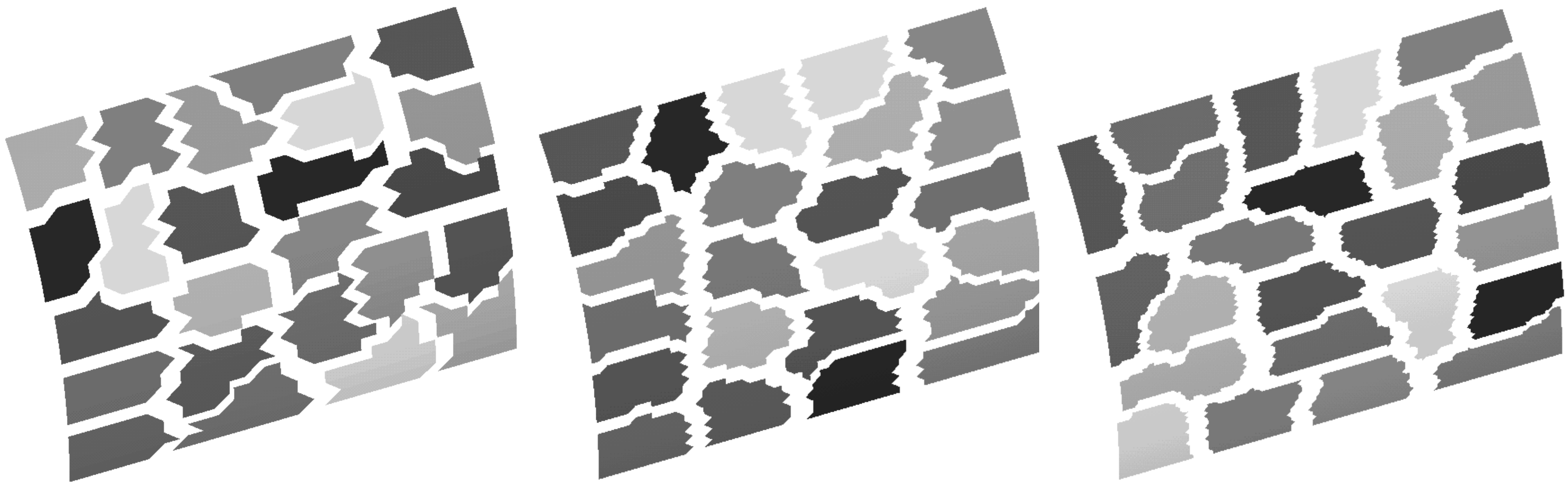
GRAVITY LOAD



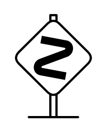



3-NODE SHELL ELEMENTS

| MESH | # ELEMENTS | # DOF |
|---|---|---|
| $h$ | 1 152 | 3 750 |
| $h/2$ | 4 608 | 14 406 |
| $h/4$ | 18 432 | 56 454 |



| MESH | INTERFACE (DOF) |
|---|---|
| $h$ | 1 182 |
| $h/2$ | 2 382 |
| $h/4$ | 4 818 |



RUGGED INTERFACE !!!

THICKNESS = 0,1 m

| | PRIMAL | | DUAL |
|---|---|---|---|
| MESH | GNN | NN+C | FETI-DP |
| $h$ | 30 iter. (726) | 42 iter. (270) | 44 iter. (270) |
| $h/2$ | 36 iter. (714) | 41 iter. (270) | 44 iter. (270) |
| $h/4$ | 39 iter. (738) | 40 iter. (276) | 45 iter. (276) |

STOPPING CRITERION

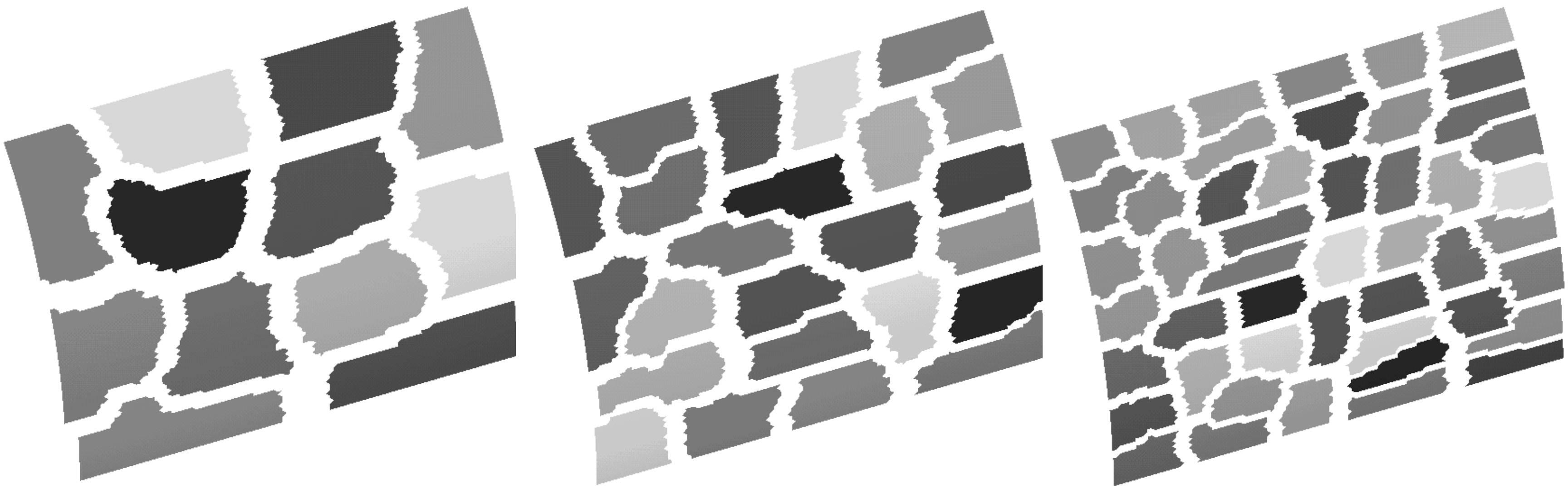
$$\frac{\| K\,u - f \|}{\| f \|} \leq 10^{-6}$$

GNN : GENERALIZED NEUMANN-NEUMANN PRECONDITIONER

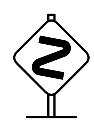
NN+C : NEUMANN-NEUMANN PRECONDITIONER + CONTINUITY AT THE CROSS-POINTS

| | | PRIMAL | | DUAL |
|---|---|---|---|---|
| MESH | THICKNESS | GNN | NN+C | FETI-DP |
| $h$ | 0,1 | 30 iter. (726) | 42 iter. (270) | 44 iter. (270) |
| | 0,01 | 31 iter. | 48 iter. | 50 iter. |
| | 0,001 | 54 iter. | 99 iter. | 106 iter. |
| $h/2$ | 0,1 | 36 iter. (714) | 41 iter. (270) | 44 iter. (270) |
| | 0,01 | 37 iter. | 48 iter. | 50 iter. |
| | 0,001 | 45 iter. | 74 iter. | 80 iter. |
| $h/4$ | 0,1 | 39 iter. (738) | 40 iter. (276) | 45 iter. (276) |
| | 0,01 | 39 iter. | 44 iter. | 47 iter. |
| | 0,001 | 42 iter. | 67 iter. | 73 iter. |

$h/4$ (56 454 DOF) - THICKNESS = 0,1 m



| $n_s$ | INTERFACE (DOF) |
|---|---|
| 12 | 3 144 |
| 24 | 4 818 |
| 48 | 7 295 |



RUGGED INTERFACE !!!

MESH $h/4$ (56 454 DOF) - THICKNESS = 0,1 m

| | PRIMAL | | DUAL |
|---|---|---|---|
| $n_s$ - (INTERFACE) | GNN | NN+C | FETI-DP |
| 12 (3 144) | 38 iter. (426) | 40 iter. (216) | 45 iter. (216) |
| 24 (4 818) | 39 iter. (738) | 40 iter. (276) | 45 iter. (276) |
| 48 (7 295) | 45 iter. (1602) | 51 iter. (630) | 56 iter. (630) |

$$\begin{cases} \text{FIND } (\lambda, q) \\ K^{-1}M\ q = \dfrac{1}{\lambda}\ q \end{cases}$$

SUBSPACE ALGORITHM

$$KZ_{k+1} = MZ_k$$

$$\Downarrow$$

SUCCESSIVE AND MULTIPLE RIGHT HAND SIDES

$$\Downarrow$$

RESTARTED CONJUGATE GRADIENT (KRYLOV SUBSPACE)

$$u_b^0 = \sum_{i=1}^{n_1+\ldots+n_{p-1}} \frac{(f_{b_p}, w_i)}{(Sw_i, w_i)}\ w_i$$

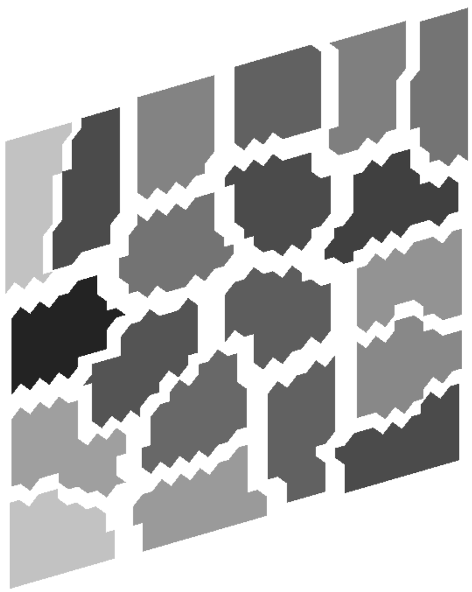
# FREE VIBRATION : PLATE (1 × 1 × 0,1 m)

2 LOWEST EIGENVALUES - SUBSPACE ALGORITHM (5 ITERATIONS)

3-NODE SHELL ELEMENTS

10 086 DOF - 20 SUBDOMAINS

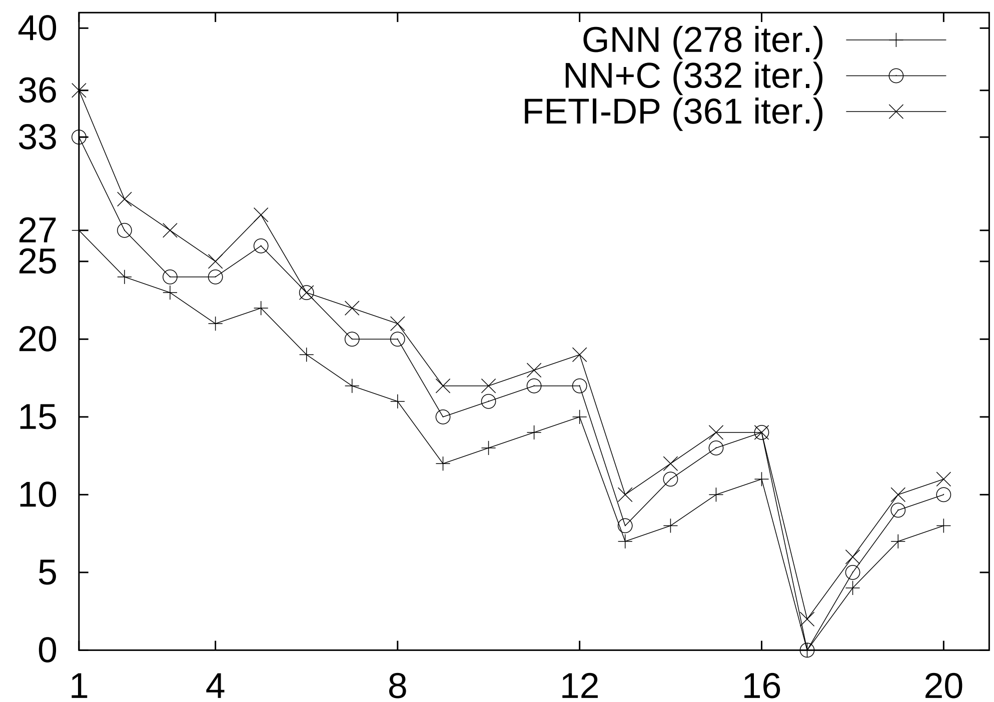
INTERFACE 1 578 DOF



COARSE PROBLEM

| | |
|---|---|
| GNN | 582 |
| NN+C, FETI-DP | 222 |

ACCUMULATION OF SEARCH DIRECTIONS



IMPLICIT NEWMARK ALGORITHM

$$\left\{ \begin{array}{l} [K + \dfrac{1}{\beta\, h^2}\ M]\, u^k = f^k \\ \text{OR} \\ [M + \beta\, h^2\ K]\, \ddot{u}^k = f^k \end{array} \right.$$

$\Downarrow$

NO RIGID BODY MODES (SHIFT)

$\Downarrow$

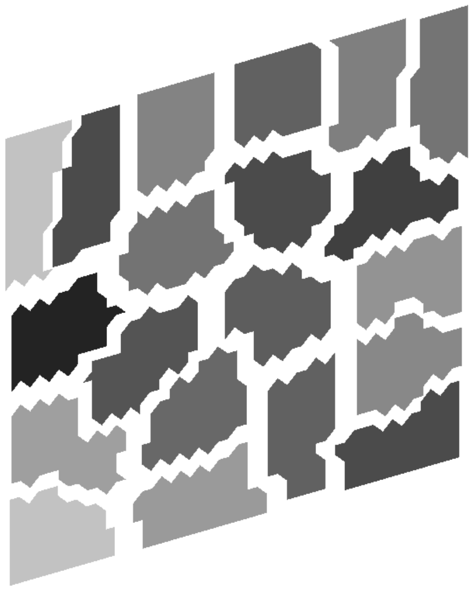
- FETI 1 OR GNN (RBM) : REINTRODUCTION OF THE RIGID BODY MODES (COSTLY)
- FETI-DP OR NN+C OR GNN (CM) : NOTHING TO DO

## IMPLICIT NEWMARK ALGORITHM

3-NODE SHELL ELEMENTS

10 086 DOF - 20 SUBDOMAINS

INTERFACE 1 578 DOF



COARSE PROBLEM

| | |
|---|---|
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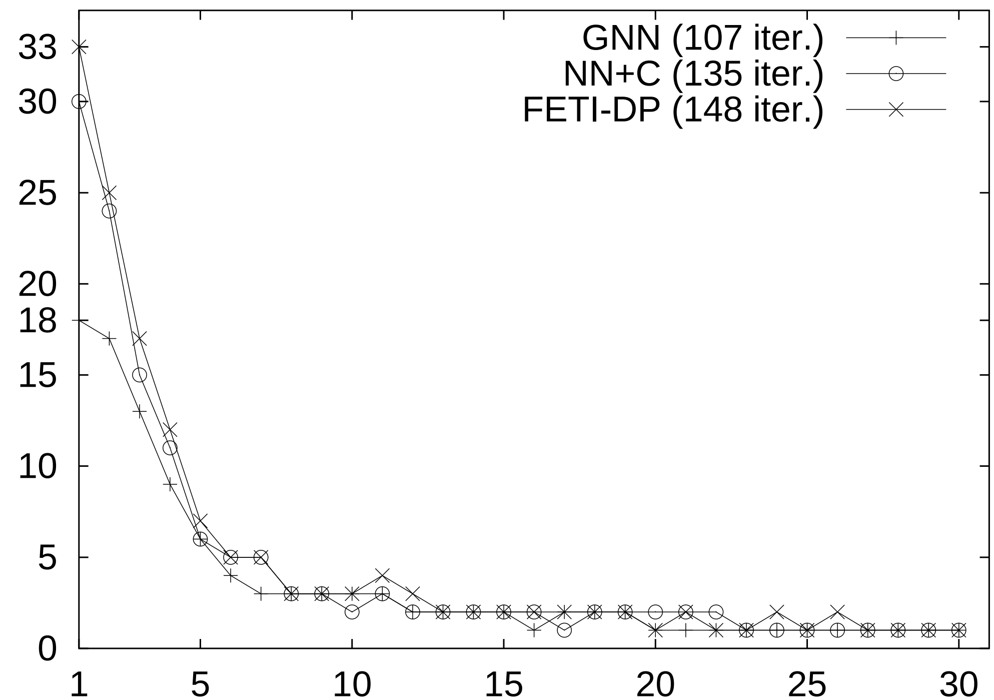
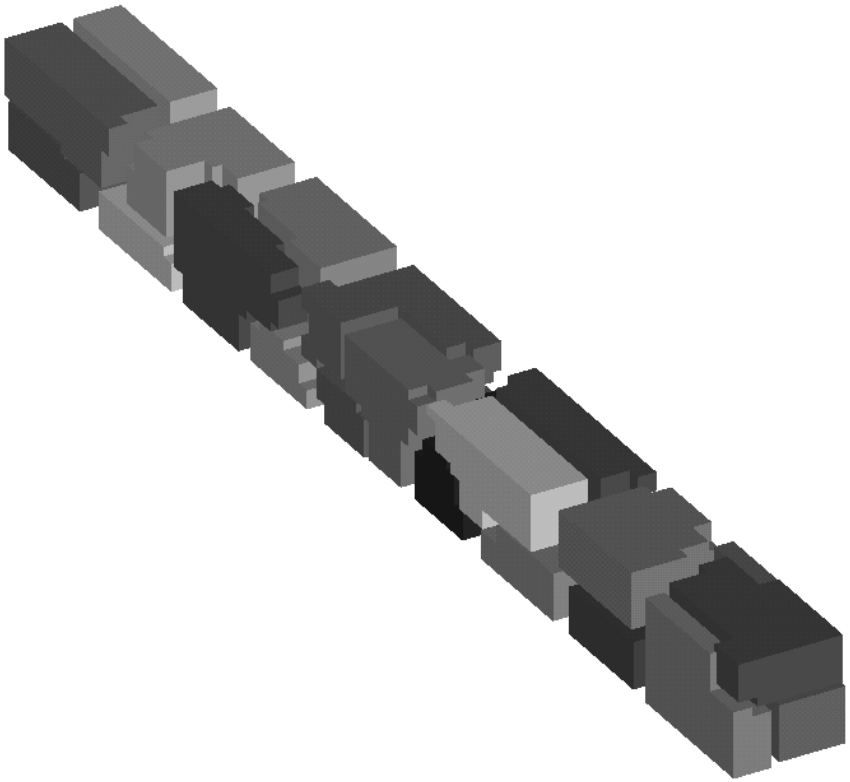
ACCUMULATION OF SEARCH DIRECTIONS

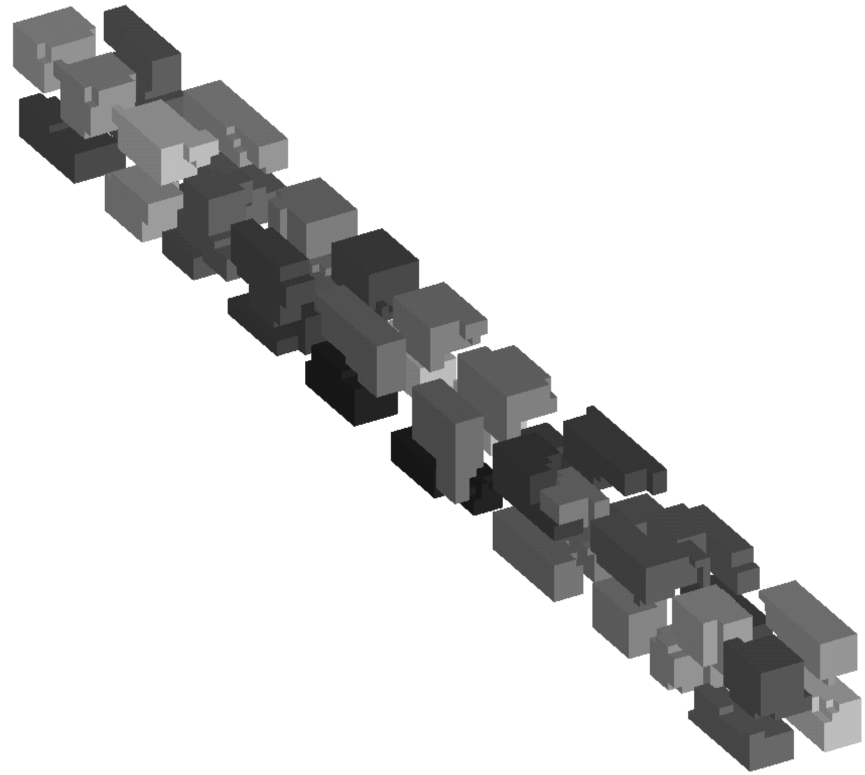


# 3D LINEAR ANALYSIS : CANTILEVER BEAM (4 $\times$ 4 $\times$ 40 m)

8-NODE BRICK ELEMENTS (12 $\times$ 12 $\times$ 76 NODES, 32 832 DOF)



20 SUBDOMAINS (INTERFACE = 7 614 DOF)



40 SUBDOMAINS (INTERFACE = 10 262 DOF)

| | PRIMAL | | DUAL | | | |
|---|---|---|---|---|---|---|
| | | | FETI-DP | | FETI-1 | |
| $n_s$ | GNN (RBM) | NN+C | DIRICHLET | LUMPED | DIRICHLET | LUMPED |
| 20 | 44 iter. | 30 iter. | 31 iter. | 60 iter. | 43 iter. | 67 iter. |
| 40 | 54 iter. | 30 iter. | 30 iter. | 53 iter. | 58 iter. | 77 iter. |

SIZE OF THE COARSE PROBLEM

| | PRIMAL | | DUAL | | | |
|---|---|---|---|---|---|---|
| | | | FETI-DP | | FETI-1 | |
| $n_s$ | GNN (RBM) | NN+C | DIRICHLET | LUMPED | DIRICHLET | LUMPED |
| 20 | 102 | 852 | | | 102 | |
| 40 | 216 | 1968 | | | 216 | |

DECREASE THE SIZE OF THE COARSE PROBLEM :

ENOUGH CROSS-POINTS TO REMOVE ALL SINGULARITIES

# CONCLUSIONS

- ❒ METHOD NUMERICALLY SCALABLE.
- ❒ PERFORMANCES ARE EQUIVALENT TO THOSE OF THE DUAL-PRIMAL FETI METHOD.
- ❒ IMPLEMENTATION EASY.

FUTURE WORKS

- ❒ PARALLEL SCALABILITY.
- ❒ VARIATION OF THE NUMBER OF DOF FIXED BY CROSS-POINTS.
- ❒ COMPARISON WITH OTHER PRECONDITIONERS.

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- ❒ PERFORMANCES ARE EQUIVALENT TO THOSE OF THE DUAL-PRIMAL FETI METHOD.
- ❒ IMPLEMENTATION EASY.
- ❒

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- ❒ PARALLEL SCALABILITY.
- ❒ VARIATION OF THE NUMBER OF DOF FIXED BY CROSS-POINTS.
- ❒ COMPARISON WITH OTHER PRECONDITIONERS.
- ❒